

A NOTE ON MODELING SOME CLASSES OF NONLINEAR SYSTEMS FROM DATA

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Abstract. We study the modeling of bilinear and quadratic systems from measured data. The measurements are given by samples of higher order frequency response functions. These values can be identified from the corresponding Volterra series of the underlying nonlinear system. We test the method for examples from structural dynamics and chemistry.

1 INTRODUCTION

With the ever-increasing supply of available measurements in the study of physical processes, the need for incorporating data in the modeling process has steadily grown. The main challenge is to use measured data to construct models that can accurately identify the underlying dynamics. Sometimes, the models have large dimensions and are not fitted for fast simulation.

Model reduction is commonly viewed as a methodology used for reducing the computational complexity of large scale complex models in numerical simulations. The goal is to construct a smaller system with the same structure and similar response characteristics as the original. For an overview of model reduction methods, we refer the reader to [1, 5].

The Loewner Framework (LF) is a data-driven identification and reduction technique that uses measured or computed data to construct reliable surrogate models. For a comprehensive tutorial paper on LF for linear systems, we refer the reader to [3]. The framework has been extended and applied to classes of mildly nonlinear systems, such as bilinear systems in [2], and quadratic-bilinear (QB) systems in [7]. These classes are of interest since nonlinear systems can be reformulated, without any approximation, as QB systems (provided that the nonlinearities are analytical).

The Volterra series is a mathematical description of the input/output relationship, valid for a wide class of nonlinear systems. It is based on a functional power series expansion. Applications of this method range in areas such as fluid dynamics, electrical engineering, mechanical engineering or biomedical engineering.

Data-driven modeling of nonlinear systems has been an active field of research over the years. However, there are still many open challenges which can not be addressed by the classical linear

identification theory. One of these issues is choosing a model structure from the many possible classes, without prior knowledge of the system.

In this work, we assume that the unknown nonlinear system to be modeled has bilinear or quadratic nonlinearities. For these classes of mild nonlinear systems, one can directly compute generalized transfer functions in the frequency domain. This is done by means of the harmonic probing method by identifying the leading terms of the Volterra series.

Using the Loewner framework, we first fit a linear model that matches samples of the first (linear) transfer function. Then, from samples of the second transfer function (that includes the nonlinear behavior), we are able to fit appropriate bilinear or quadratic terms. The data required for this procedure can be estimated from direct numerical simulations.

The paper is structured as follows; after the introduction in Section 1, we give a brief overview on the Volterra series in Section 2. Then, Section 3 introduces linear systems and an overview of the Loewner framework. Afterward, Sections 4 and 5 include the derivations of transfer function for bilinear, and respectively, for quadratic systems. The numerical examples are presented Section 6, while Section 7 concludes the paper.

2 VOLTERRA SERIES

The Volterra series theory describes the relationship between the control input and the observed output of a dynamical system whose dynamics is characterized by nonlinear behavior.

It provides an explicit formulation through the so-called higher-order Volterra kernels. In time-domain, the mappings are given by the higher-order impulse response functions (HOIRFs), while in the frequency domain, we have the higher-order frequency response functions (HOFRFs). The latter quantities can be derived from the HOIRFs by applying a generalized Fourier or Laplace transformation. A method for deriving the analytical expressions of the HOFRFs was proposed in [4]. This has since been referred to as the harmonic (input) probing method for continuous-time nonlinear systems. It is based on the fact that a harmonic input must result in a harmonic output.

Considerable amount of work has been employed into this direction in the 80's. We mention the books [11, 12] which represent exhaustive collections of theoretical results related to the Volterra series. Afterwards, contributions were made on applications in certain fields such as structural dynamics (see [14]), and in signal processing (see [6]).

The input-output relationship for a wide class of nonlinear systems can be expressed as a Volterra series (here $\mathbf{u}(t)$ is the input while $\mathbf{y}(t)$ the output)

$$\mathbf{y}(t) = \sum_{n=1}^{\infty} \mathbf{y}_n(t), \quad \mathbf{y}_n(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathbf{h}_n(\tau_1, \dots, \tau_n) \prod_{i=1}^n \mathbf{u}(t - \tau_i) d\tau_i. \quad (1)$$

In (1), the multivariate function $\mathbf{h}_n(\tau_1, \dots, \tau_n)$ is the n th-order Volterra kernel or generalized impulse response function of order n . The frequency domain description of this kernel can be found by applying a generalized multidimensional Fourier transform. Hence, one can write

$$\mathbf{H}_n(j\omega_1, \dots, j\omega_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathbf{h}_n(\tau_1, \dots, \tau_n) e^{-j(\omega_1\tau_1 + \cdots + \omega_n\tau_n)} d\tau_1 \cdots d\tau_n. \quad (2)$$

In (2), the multivariate function $\mathbf{H}_n(j\omega_1, \dots, j\omega_n)$ is the n th-order generalized transfer function. For example, if $n = 1$, then \mathbf{H}_1 is the linear transfer function derived in Section 3.

The harmonic (input) probing method consists in applying periodic functions as input in the following format

$$\mathbf{u}(t) = \sum_{\ell=1}^p \alpha_{\ell} e^{j\omega_{\ell} t}. \quad (3)$$

Then, by replacing this input into (1), one can write the n th element of the output, i.e. \mathbf{y}_n as

$$\mathbf{y}_n(t) = \sum_{\ell_1=1}^p \cdots \sum_{\ell_n=1}^p \alpha_{\ell_1} \cdots \alpha_{\ell_n} \mathbf{H}_n(j\omega_{\ell_1}, \dots, j\omega_{\ell_n}) e^{j(\omega_{\ell_1} + \cdots + \omega_{\ell_n})t}. \quad (4)$$

For simplicity, in Sections 4 and 5, we use a single harmonic function ($p = 1$ and $\mathbf{u}(t) = \alpha e^{j\omega t}$) to identify explicit formulas for the generalized transfer functions (in the case of bilinear and quadratic systems). Consequently, the transfer function \mathbf{H}_n will be evaluated at multiple instances of the driving frequency $j\omega$.

3 LINEAR SYSTEMS

A linear system Σ_L is characterized by a set of differential equations, as follows

$$\Sigma_L: \quad \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \quad (5)$$

where $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B}, \mathbf{C}^T \in \mathbb{R}^n$. The matrix \mathbf{E} is considered to be invertible and hence assimilated into matrices \mathbf{A} and \mathbf{B} : $\bar{\mathbf{A}} = \mathbf{E}^{-1}\mathbf{A}$, $\bar{\mathbf{B}} = \mathbf{E}^{-1}\mathbf{B}$. One can explicitly write the output $\mathbf{y}(t)$ in terms of the input $\mathbf{u}(t)$:

$$\mathbf{y}(t) = \mathbf{C}e^{\bar{\mathbf{A}}t}\mathbf{x}(0) + \int_0^t \mathbf{C}e^{\bar{\mathbf{A}}(t-\tau)}\bar{\mathbf{A}}\mathbf{u}(\tau)d\tau, \quad \tau \in [0, t]. \quad (6)$$

The linear time-domain kernel is defined as follows $\mathbf{h}(t) = \mathbf{C}e^{\bar{\mathbf{A}}t}\bar{\mathbf{B}}$ and it is referred to as the impulse response. By considering the Laplace transform of this function, we obtain:

$$\mathbf{H}(s) = \mathcal{L}\{\mathbf{h}(t)\} = \mathbf{C}(s\mathbf{I} - \bar{\mathbf{A}})^{-1}\bar{\mathbf{B}}. \quad (7)$$

Alternatively, simply apply the Laplace transform directly to the differential equation in (5)

$$s\mathbf{E}\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \Rightarrow \mathbf{X}(s) = (s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \Rightarrow \mathbf{Y}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \quad (8)$$

Hence, identify the transfer function as the ratio of the transformed output $\mathbf{Y}(s)$ to the transformed input $\mathbf{U}(s)$:

$$\mathbf{H}(s) = \frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}. \quad (9)$$

3.1 The Loewner framework

The Loewner framework is a data-driven model identification and reduction tool (see [3]). It provides a solution through rational approximation and by means of interpolation. The problem can be formulated as follows; given the following data

$$\text{right data : } (\lambda_i; \mathbf{w}_i), \quad i = 1, \dots, k, \quad \text{and} \quad \text{left data : } (\mu_j; \mathbf{v}_j), \quad j = 1, \dots, q. \quad (10)$$

find the function $\mathbf{H}(s)$, such that the following interpolation conditions are fulfilled:

$$\mathbf{H}(\lambda_i) = \mathbf{w}_i, \quad \mathbf{H}(\mu_j) = \mathbf{v}_j, \quad (11)$$

where $\mathbf{w}_i, \mathbf{v}_j \in \mathbb{C}$, could be, for instance, S-parameters, i.e. values that could be measured experimentally. Proceed to arranging the data (10) into matrix format. The Loewner matrix $\mathbb{L} \in \mathbb{C}^{k \times k}$ and the shifted Loewner matrix $\mathbb{L}_s \in \mathbb{C}^{k \times k}$ are defined as follows

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1 - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1 - \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_k - \mathbf{w}_1}{\mu_k - \lambda_1} & \cdots & \frac{\mathbf{v}_k - \mathbf{w}_k}{\mu_k - \lambda_k} \end{bmatrix}, \quad \mathbb{L}_s = \begin{bmatrix} \frac{\mu_1 \mathbf{v}_1 - \mathbf{w}_1 \lambda_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mu_1 \mathbf{v}_1 - \mathbf{w}_k \lambda_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mu_k \mathbf{v}_k - \mathbf{w}_1 \lambda_1}{\mu_k - \lambda_1} & \cdots & \frac{\mu_k \mathbf{v}_k - \mathbf{w}_k \lambda_k}{\mu_k - \lambda_k} \end{bmatrix}, \quad (12)$$

while the data vectors $\mathbf{V}, \mathbf{W}^T \in \mathbb{R}^k$ are introduced below

$$\mathbf{V} = [\mathbf{v}_1; \mathbf{v}_2; \dots; \mathbf{v}_q], \quad \mathbf{W} = [\mathbf{w}_1 \quad \mathbf{w}_2 \quad \cdots \quad \mathbf{w}_k]. \quad (13)$$

Lemma 1. *If the matrix pencil $(\mathbb{L}_s, \mathbb{L})$ is regular, then one can directly recover the system matrices and the interpolation function. The Loewner model is composed of*

$$\mathbf{E} = -\mathbb{L}, \quad \mathbf{A} = -\mathbb{L}_s, \quad \mathbf{B} = \mathbf{V}, \quad \mathbf{C} = \mathbf{W}, \quad \rightarrow \quad \mathbf{H}(s) = \mathbf{W}(\mathbb{L}_s - s\mathbb{L})^{-1}\mathbf{V}. \quad (14)$$

Introduce \mathcal{R} as the generalized controllability matrix in terms of the right sampling points $\{\lambda_1, \dots, \lambda_k\}$ and matrices $\mathbf{E}, \mathbf{A}, \mathbf{B}$. Additionally, let \mathcal{O} be the generalized observability matrix, written in terms of the left sampling points $\{\mu_1, \dots, \mu_k\}$ and matrices $\mathbf{E}, \mathbf{A}, \mathbf{C}$.

$$\mathcal{O} = \begin{bmatrix} \mathbf{C}(\mu_1 \mathbf{E} - \mathbf{A})^{-1} \\ \vdots \\ \mathbf{C}(\mu_k \mathbf{E} - \mathbf{A})^{-1} \end{bmatrix}, \quad \mathcal{R} = [(\lambda_1 \mathbf{E} - \mathbf{A})^{-1} \mathbf{B} \quad \cdots \quad (\lambda_k \mathbf{E} - \mathbf{A})^{-1} \mathbf{B}]. \quad (15)$$

Then, it follows that the data matrices can be factorized as follows

$$\mathbb{L} = -\mathcal{O}\mathbf{E}\mathcal{R}, \quad \mathbb{L}_s = -\mathcal{O}\mathbf{A}\mathcal{R}, \quad \mathbf{V} = \mathbf{C}\mathcal{R}, \quad \mathbf{W} = \mathcal{O}\mathbf{B}. \quad (16)$$

In practical applications, the pencil $(\mathbb{L}_s, \mathbb{L})$ is often singular. In these cases, perform a rank revealing singular value decomposition (SVD) of the Loewner matrix \mathbb{L} . By setting $\text{rank } \mathbb{L} = k$, write (where $\mathbf{X}_r = \mathbf{X}(:, 1:r)$, $\mathbf{Y}_r = \mathbf{Y}(:, 1:r)$, and $\mathbf{S}_r = \mathbf{S}(1:r, 1:r)$)

$$\mathbb{L} = \mathbf{X}\mathbf{S}\mathbf{Y}^* \approx \mathbf{X}_r\mathbf{S}_r\mathbf{Y}_r^*, \quad \text{with } \mathbf{X}_r, \mathbf{Y}_r \in \mathbb{C}^{k \times r}, \quad \mathbf{S}_r \in \mathbb{C}^{r \times r}. \quad (17)$$

Lemma 2. *The system matrices corresponding to a compressed /projected Loewner model for which the transfer function approximately matches the conditions in (11), can be computed as*

$$\hat{\mathbf{E}} = -\mathbf{X}_r^* \mathbb{L} \mathbf{Y}_r, \quad \hat{\mathbf{A}} = -\mathbf{X}_r^* \mathbb{L}_s \mathbf{Y}_r, \quad \hat{\mathbf{B}} = \mathbf{X}_r^* \mathbf{V}, \quad \hat{\mathbf{C}} = \mathbf{W} \mathbf{Y}_r. \quad (18)$$

and the corresponding interpolating function $\hat{\mathbf{H}}(s) = \mathbf{W} \mathbf{Y}_r (\mathbf{X}_r^* (\mathbb{L}_s - s\mathbb{L}) \mathbf{Y}_r)^{-1} \mathbf{X}_r^* \mathbf{V}$.

Remark 3.1. *One can incorporate the reduced-order matrix $\hat{\mathbf{E}}$ in (18) into the other system matrices and, in this way, put together a model $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}})$ in standard representation:*

$$\tilde{\mathbf{A}} = \hat{\mathbf{E}}^{-1} \hat{\mathbf{A}} = (\mathbf{X}_r^* \mathbb{L} \mathbf{Y}_r)^{-1} (\mathbf{X}_r^* \mathbb{L}_s \mathbf{Y}_r), \quad \tilde{\mathbf{B}} = \hat{\mathbf{E}}^{-1} \hat{\mathbf{B}} = (\mathbf{X}_r^* \mathbb{L} \mathbf{Y}_r)^{-1} (\mathbf{X}_r^* \mathbf{V}), \quad \tilde{\mathbf{C}} = \hat{\mathbf{C}} = \mathbf{W} \mathbf{Y}_r. \quad (19)$$

Definition 3.1. The Kronecker product of two vectors $\mathbf{x} \in \mathbb{C}^m$ and $\mathbf{y} \in \mathbb{C}^m$ is given by the vector $\mathbf{z} \in \mathbb{C}^{mn}$ defined in the following way

$$\mathbf{z} = \mathbf{x} \otimes \mathbf{y} = \begin{bmatrix} x_1 y_1, & x_1 y_2, & \dots & x_1 y_m, & x_2 y_1, & \dots, & x_2 y_m, & \dots, & x_n y_1, & \dots, & x_n y_m \end{bmatrix}^T.$$

Note that the product is not commutative. The definition of the product for matrices follows in an analogous way.

4 BILINEAR SYSTEMS

We analyze bilinear systems Σ_B of dimension n described by

$$\Sigma_B : \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{N}\mathbf{x}\mathbf{u} + \mathbf{B}\mathbf{u}, \quad \mathbf{y} = \mathbf{C}\mathbf{u}. \quad (20)$$

where $\mathbf{A}, \mathbf{N} \in \mathbb{R}^{n \times n}$ and $\mathbf{B}, \mathbf{C}^T \in \mathbb{R}^n$. The nonlinearity, although mild, is given by the product of the variable \mathbf{x} with the input \mathbf{u} in (20) (and scaled by matrix \mathbf{N}). We introduce a numerical tool that allows approximating nonlinear systems by means of a bilinear system. The technique is commonly known as Carleman linearization (see [11]).

This procedure is a linearization of the initial nonlinear system and it involves approximation techniques (Taylor series expansion and truncation) to yield a bilinear system. We demonstrate the method through an example.

Example 4.1. Consider the following non-linear scalar system characterized by the equations:

$$\Sigma_N^1 : \dot{x}(t) = -2x(t) + \sin(x(t)) + u(t), \quad y(t) = x(t). \quad (21)$$

Write the Taylor series around 0 for $\sin x$, where $c_{2k+1} = \frac{(-1)^k}{(2k+1)!}$ and $c_{2k} = 0$, as

$$\sin(x(t)) = \sum_{k=1}^{\infty} c_k x^k(t) = x(t) - \frac{x^3(t)}{3!} + \dots, \Rightarrow \dot{x}(t) = -x(t) - \sum_{k=3}^{\infty} c_k x^k(t) + u(t).$$

Truncate at $N = 3$ and assemble the new state vector $\hat{\mathbf{x}} = [x, x^2, x^3]^T \in \mathbb{R}^3$. Rewrite the equations in (21) as a bilinear system, exactly as in (20)

$$\begin{aligned} \dot{\hat{\mathbf{x}}}(t) &= \mathbf{A}_1 \hat{\mathbf{x}}(t) + \mathbf{N}_1 \hat{\mathbf{x}}(t)u(t) + \mathbf{B}_1 u(t), \quad y(t) = \mathbf{C}_1 \hat{\mathbf{x}}(t), \\ \mathbf{A}_1 &= \begin{bmatrix} -1 & 0 & -\frac{1}{3!} \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad \mathbf{N}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{C}_1 = [1 \ 0 \ 0]. \end{aligned}$$

4.1 Deriving transfer functions

We choose the harmonic input signal $\mathbf{u}(t) = \alpha e^{j\omega t}$, $\omega, \alpha > 0$. Next, as in [12], assume that the solution of (20) is written as an infinite power series

$$\mathbf{x}(t) = \sum_{m=1}^{\infty} \mathbf{G}_m(j\omega, \dots, j\omega) \alpha^m e^{mj\omega t}. \quad (22)$$

Consider the DC term to be 0 ($\mathbf{G}_0 = 0$). By substituting the explicit solution formula (3) into the original differential equation (1), it follows that:

$$\begin{aligned} \sum_{m=1}^{\infty} mj\omega \mathbf{G}_m(j\omega, \dots, j\omega) \alpha^m e^{mj\omega t} &= \mathbf{A} \left(\sum_{m=1}^{\infty} \mathbf{G}_m(j\omega, \dots, j\omega) \alpha^m e^{mj\omega t} \right) + \\ &\quad \mathbf{N} \left(\sum_{m=1}^{\infty} \mathbf{G}_m(j\omega, \dots, j\omega) \alpha^m e^{mj\omega t} \right) \alpha e^{j\omega t} + \mathbf{B} \alpha e^{j\omega t}. \end{aligned} \quad (23)$$

In what follows, we refer to $\mathbf{G}_m(j\omega, \dots, j\omega)$ by using the condensed notation $\mathbf{G}_m(j\omega)$ (similarly, for the transfer function $\mathbf{H}_m(j\omega, \dots, j\omega)$ use the notation $\mathbf{H}_m(j\omega)$ instead).

Initiate the identification procedure by equating the coefficient of the term $e^{mj\omega t}$, $\forall m \geq 1$ from both left and right sides of equation (6).

1). **The case $m = 1 \rightarrow$ coefficients of $e^{j\omega t}$.**

$$j\omega \mathbf{G}_1(j\omega)\alpha = \mathbf{A}\mathbf{G}_1(j\omega)\alpha + \mathbf{B}\alpha \Rightarrow \mathbf{G}_1(j\omega) = (j\omega\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \Phi(j\omega)\mathbf{B}, \quad (24)$$

where $\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1}$. Hence, write the first (linear) transfer function as

$$\mathbf{H}_1(j\omega) = \mathbf{C}\Phi(j\omega)\mathbf{B}. \quad (25)$$

2). **The case $m = 2 \rightarrow$ coefficients of $e^{2j\omega t}$.**

$$2j\omega \mathbf{G}_2(j\omega)\alpha^2 = \mathbf{A}\mathbf{G}_2(j\omega, j\omega)\alpha^2 + \mathbf{N}\mathbf{G}_1(j\omega)\alpha^2 \Rightarrow \mathbf{G}_2(j\omega) = \Phi(2j\omega)\mathbf{N}\Phi(j\omega)\mathbf{B}.$$

Hence, write the second transfer function as

$$\mathbf{H}_2(j\omega) = \mathbf{C}\Phi(2j\omega)\mathbf{N}\Phi(j\omega)\mathbf{B}. \quad (26)$$

n). **The general case \rightarrow coefficients of $e^{nj\omega t}$.**

$$nj\omega \mathbf{G}_n(j\omega)\alpha^n = \mathbf{A}\mathbf{G}_n(j\omega)\alpha^n + \mathbf{N}\mathbf{G}_{n-1}(j\omega)\alpha^n \Rightarrow (nj\omega\mathbf{I} - \mathbf{A})\mathbf{G}_n(j\omega) = \mathbf{N}\mathbf{G}_{n-1}(j\omega) \quad (27)$$

Hence, one can write $\mathbf{G}_n(j\omega) = \Phi(nj\omega)\mathbf{N}\mathbf{G}_{n-1}(j\omega)$. In order to derive the n th transfer function $\mathbf{H}_n(j\omega)$, just multiply $\mathbf{G}_n(j\omega)$ to the left with \mathbf{C} , i.e.

$$\mathbf{H}_n(j\omega) = \mathbf{C}\Phi(nj\omega)\mathbf{N}\Phi((n-1)j\omega)\mathbf{N}\cdots\mathbf{N}\Phi(j\omega)\mathbf{B}. \quad (28)$$

4.2 Estimating the bilinear term \mathbf{N}

As explained in Section 3.1, the first step is to construct a reliable reduced-order linear model $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ of order r as in (19) for the data in (10). This set is composed of samples of the first transfer function $\mathbf{H}_1(j\omega)$ in (25).

The next step is to fit an appropriate matrix $\mathbf{N} \in \mathbb{C}^{r \times r}$ that supplements the linear model as bilinear model. In this direction, it is assumed that information about the second transfer function $\mathbf{H}_2(j\omega)$ in (26) is known at $2k$ points $\{j\omega_1, \dots, j\omega_{2k}\}$. Introduce the following vectors $\mathbf{r}_\ell \in \mathbb{C}^r$ and $\mathbf{o}_\ell \in \mathbb{C}^{1 \times r}$ for $\ell \in \{1, 2, \dots, 2k\}$:

$$\mathbf{r}_\ell = (j\omega_\ell\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \Phi(j\omega_\ell)\mathbf{B}, \quad \mathbf{o}_\ell = \mathbf{C}(2j\omega_\ell\mathbf{I} - \mathbf{A})^{-1} = \mathbf{C}\Phi(2j\omega_\ell). \quad (29)$$

Note that one can write the matrices \mathcal{O} and \mathcal{R} in (15) in terms of the new vectors. For the set of points $\{2j\omega_1, \dots, 2j\omega_{2k}\}$, write the observability matrix as $\mathcal{O} = [\mathbf{o}_1; \mathbf{o}_2; \dots; \mathbf{o}_{2k}]$. Then write the controllability matrix as $\mathcal{R} = [\mathbf{r}_1 \ \mathbf{r}_2 \ \cdots \ \mathbf{r}_{2k}]$ for the set of points $\{j\omega_1, \dots, j\omega_{2k}\}$.

One can write the second transfer function evaluated at $j\omega_\ell$, for $\ell \in \{1, 2, \dots, 2k\}$, as

$$\mathbf{H}_2(j\omega_\ell) = \mathbf{C}\Phi(2j\omega_\ell)\mathbf{N}\Phi(j\omega_\ell)\mathbf{B} = \mathbf{o}_\ell\mathbf{N}\mathbf{r}_\ell \quad (30)$$

Denote with $\mathbf{v}_\mathbf{N} \in \mathbb{C}^{r^2}$ the vectorization of $\mathbf{N} \in \mathbb{C}^{r \times r}$, i.e. $\mathbf{v}_\mathbf{N} = [\mathbf{N}(:, 1); \mathbf{N}(:, 2) \ \cdots \ \mathbf{N}(:, r)]$. From (30), we can write that $(\mathbf{r}_\ell^T \otimes \mathbf{o}_\ell)\mathbf{v}_\mathbf{N} = \mathbf{H}_2(j\omega_\ell)$, $\forall 1 \leq \ell \leq 2k$. By collecting this $2k$ equalities into a matrix format, we put together the following linear equation:

$$\mathbf{Z}\mathbf{v}_\mathbf{N} = \mathbf{V}_2, \quad \text{where } \mathbf{Z} \in \mathbb{C}^{2k \times r^2}, \ \mathbf{V}_2 \in \mathbb{C}^{2k}, \quad \begin{cases} \mathbf{Z}(\ell, :) = \mathbf{r}_\ell^T \otimes \mathbf{o}_\ell \\ \mathbf{V}_2(\ell) = \mathbf{H}_2(j\omega_\ell) \end{cases}. \quad (31)$$

One can write the solution of (31) by means of the Moore-Penrose pseudo-inverse of the matrix \mathbf{Z} which is denoted with $\mathbf{Z}^\dagger \in \mathbb{C}^{r^2 \times 2k}$, as

$$\mathbf{v}_\mathbf{N} = \mathbf{Z}^\dagger\mathbf{V}_2. \quad (32)$$

Finally, simply put together the recovered matrix $\mathbf{N} \in \mathbb{C}^{r \times r^2}$ from $\mathbf{v}_\mathbf{N}$.

5 QUADRATIC SYSTEMS

We analyze quadratic systems Σ_Q of dimension n described by

$$\Sigma_Q : \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{Q}(\mathbf{x} \otimes \mathbf{x}) + \mathbf{B}\mathbf{u}, \quad \mathbf{y} = \mathbf{C}\mathbf{u}. \quad (33)$$

where $\mathbf{Q} \in \mathbb{C}^{n \times n^2}$ and \mathbf{A}, \mathbf{B} and \mathbf{C} are as in (5). We introduce a transformation which has been known in the literature as the McCormick relaxation (see [9]). In broad terms, this result states that any nonlinear system characterized by elementary functions can be equivalently recast as a quadratic system (with a possible bilinear term). This result applies, for example, to nonlinearities such as polynomial, rational, exponential, logarithmic, trigonometric, or a composition thereof. We again demonstrate the method through an example.

Example 5.1. Consider the same system as in Example 4.1, but without the control u i.e

$$\Sigma_N^2 : \dot{x} = -2x + \sin x, \quad y = x.$$

Introduce two additional variables: $z = \sin x$ and $w = \cos x$, and compute the time derivatives and then substitute the given terms

$$\dot{x} = -2x + z, \quad \dot{z} = \dot{x} \cos x = -2wx + wz, \quad \dot{w} = -\dot{x} \sin x = 2zx - z^2.$$

Let $\tilde{\mathbf{x}} = [x, z, w]^T$ be the new variable; recast the original system as a quadratic system

$$\dot{\tilde{\mathbf{x}}} = \underbrace{\begin{bmatrix} -2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{A}_2} \tilde{\mathbf{x}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{Q}_2} (\tilde{\mathbf{x}} \otimes \tilde{\mathbf{x}}), \quad y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{\mathbf{C}_2} \tilde{\mathbf{x}}.$$

5.1 Deriving transfer functions

Consider the control input signal as for the bilinear case $\mathbf{u}(t) = \alpha e^{j\omega t}$, $\omega, \alpha > 0$ and make the same assumption for the solution of (33), i.e.

$$\mathbf{x}(t) = \sum_{m=1}^{\infty} \mathbf{G}_m(j\omega) \alpha^m e^{mj\omega t}. \quad (34)$$

Substitute the relation (34) into the original differential equation (33), and hence write that:

$$\begin{aligned} \sum_{m=1}^{\infty} mj\omega \mathbf{G}_m(j\omega) \alpha^m e^{mj\omega t} &= \mathbf{A} \sum_{m=1}^{\infty} \mathbf{G}_m(j\omega) \alpha^m e^{mj\omega t} + \\ \mathbf{Q} \left(\sum_{m=1}^{\infty} \mathbf{G}_m(j\omega) \alpha^m e^{mj\omega t} \right) \otimes \left(\sum_{m=1}^{\infty} \mathbf{G}_m(j\omega) \alpha^m e^{mj\omega t} \right) &+ \mathbf{B} \alpha e^{j\omega t}. \end{aligned} \quad (35)$$

Next, as for the bilinear case, equate the coefficient of the term $e^{mj\omega t}$, $\forall m \geq 1$ from both left and right sides of equation (35). The first transfer function is identified as $\mathbf{H}_1(j\omega) = \mathbf{C}\Phi(j\omega)\mathbf{B}$.

2). **The case $m = 2 \rightarrow$ coefficients of $e^{2j\omega t}$.**

$$\begin{aligned} 2j\omega \mathbf{G}_2(j\omega) \alpha^2 &= \mathbf{A} \mathbf{G}_2(j\omega) \alpha^2 + \mathbf{Q}[\mathbf{G}_1(j\omega) \otimes \mathbf{G}_1(j\omega)] \alpha^2, \\ \Rightarrow \mathbf{G}_2(j\omega) &= \Phi(2j\omega) \mathbf{Q}[\Phi(j\omega) \mathbf{B} \otimes \Phi(j\omega) \mathbf{B}]. \end{aligned}$$

Hence, write the second transfer function as

$$\mathbf{H}_2(j\omega) = \mathbf{C}\Phi(2j\omega)\mathbf{Q}\left[\Phi(j\omega)\mathbf{B} \otimes \Phi(j\omega)\mathbf{B}\right]. \quad (36)$$

n). **The general case \rightarrow coefficients of $e^{nj\omega t}$.**

$$\begin{aligned} nj\omega \mathbf{G}_n(j\omega)\alpha^n &= \mathbf{A}\mathbf{G}_n(j\omega)\alpha^n + \mathbf{Q}[\mathbf{G}_{n-1}(j\omega) \otimes \mathbf{G}_1(j\omega) + \dots + \mathbf{G}_1(j\omega) \otimes \mathbf{G}_{n-1}(j\omega)]\alpha^n. \\ \Rightarrow (nj\omega \mathbf{I} - \mathbf{A})\mathbf{G}_n(j\omega) &= \mathbf{Q}\left[\sum_{\ell=1}^{n-1} \mathbf{G}_\ell(j\omega) \otimes \mathbf{G}_{n-\ell}(j\omega)\right]. \end{aligned} \quad (37)$$

Hence identify $\mathbf{G}_n(j\omega)$ and by multiplying it to the left with \mathbf{C} , we get

$$\mathbf{H}_n(j\omega) = \mathbf{C}\Phi(nj\omega)\mathbf{Q}\left[\sum_{\ell=1}^{n-1} \mathbf{G}_\ell(j\omega) \otimes \mathbf{G}_{n-\ell}(j\omega)\right], \quad \forall n \geq 1. \quad (38)$$

5.2 Estimating the quadratic term \mathbf{Q}

The next step is to fit an appropriate matrix $\mathbf{Q} \in \mathbb{C}^{r \times r^2}$ for the quadratic model. It is assumed that information about the second transfer function $\mathbf{H}_2(j\omega)$ in (36) is known at $2k$ points $\{j\omega_1, \dots, j\omega_{2k}\}$.

One can write the second transfer function evaluated at $j\omega_\ell$, $\ell \in \{1, 2, \dots, 2k\}$, in terms of the vectors $\mathbf{r}, \mathbf{o}^* \in \mathbb{C}^r$ in (29), as follows

$$\mathbf{H}_2(j\omega_\ell) = \mathbf{C}(2j\omega_\ell \mathbf{I} - \mathbf{A})^{-1}\mathbf{Q}\left[(j\omega_\ell \mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \otimes (j\omega_\ell \mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\right] = \mathbf{o}_\ell \mathbf{Q}[\mathbf{r}_\ell \otimes \mathbf{r}_\ell] \quad (39)$$

Denote with $\mathbf{v}_\mathbf{Q} \in \mathbb{C}^{r^3}$ the vectorization of $\mathbf{Q} \in \mathbb{C}^{r \times r^2}$, i.e. $\mathbf{v}_\mathbf{Q} = [\mathbf{Q}(:, 1); \mathbf{Q}(:, 2) \dots \mathbf{Q}(:, r^2)]$. From (39), we can write that $(\mathbf{r}_\ell^T \otimes \mathbf{r}_\ell^T \otimes \mathbf{o}_\ell)\mathbf{v}_\mathbf{Q} = \mathbf{H}_2(j\omega_\ell)$, $\forall 1 \leq \ell \leq 2k$. By collecting this $2k$ equalities into a matrix format, we put together the following linear equation:

$$\mathbf{T}\mathbf{v}_\mathbf{Q} = \mathbf{V}_2, \quad \text{where } \mathbf{T} \in \mathbb{C}^{2k \times r^3}, \quad \mathbf{V}_2 \in \mathbb{C}^{2k}, \quad \begin{cases} \mathbf{T}(\ell, :) = \mathbf{r}_\ell^T \otimes \mathbf{r}_\ell^T \otimes \mathbf{o}_\ell \\ \mathbf{V}_2(\ell) = \mathbf{H}_2(j\omega_\ell) \end{cases}. \quad (40)$$

Finally, one can write the solution of (40) by means of $\mathbf{T}^\dagger \in \mathbb{C}^{r^3 \times 2k}$ as in Section 4.2, i.e. $\mathbf{v}_\mathbf{Q} = \mathbf{T}^\dagger \mathbf{V}_2$. Then one can directly put together the recovered matrix $\mathbf{Q} \in \mathbb{C}^{r \times r^2}$.

6 NUMERICAL EXAMPLES

6.1 The Langmuir kinetics model

The first example to be analyzed is given by the Langmuir kinetic equation. We use the same model as in [10]. There, this example is presented as a simple kinetic case for which the adsorption process is governed by the rates of adsorption and desorption of the solute molecules onto and from the surface. The Langmuir kinetic equation is given as follows

$$\dot{Q}(t) = k_a C(t)[Q_0 - Q(t)] - k_d Q(t), \quad (41)$$

where C is the concentration in the gas phase, Q the concentration in the solid phase and Q_0 the concentration in the solid phase corresponding to maximal coverage. Finally, k_a and k_d are the adsorption and, respectively, the desorption rate constant.

Equation (41) can be rewritten as in [10] by using instead variables defined as dimensionless deviations from a stationary state

$$\dot{q}(t) = k'_a c(t)[q_0 - q(t)] - k'_d q(t), \quad (42)$$

where the new quantities can be derived as follows

$$c = \frac{C - C_s}{C_s}, \quad q = \frac{Q - Q_s}{Q_s}, \quad q_0 = \frac{Q_0 - Q_s}{Q_s}, \quad k'_a = k_a C_s, \quad k'_d = k_d + k_a C_s. \quad (43)$$

Note that, because of the term $c(t)q(t)$ in (42), the adsorption process is nonlinear. Since $q(t)$ is the variable and $c(t)$ the input, the system is bilinear. In [10], the first three transfer functions are explicitly derived.

By choosing the input of the system to be $c(t) = \alpha e^{j\omega t}$, the functions are identified as follows:

$$\begin{aligned} \mathbf{H}_1(j\omega) &= \frac{K q_0}{j\omega\eta + 1}, \quad \mathbf{H}_2(j\omega) = -\frac{1}{q_0} \mathbf{H}_1(2j\omega) \mathbf{H}_1(j\omega), \\ \mathbf{H}_3(j\omega) &= \frac{1}{q_0^2} \mathbf{H}_1(3j\omega) \mathbf{H}_1(2j\omega) \mathbf{H}_1(j\omega), \end{aligned} \quad (44)$$

where $K = \frac{k'_a}{k'_d}$ and $\eta = \frac{1}{k'_d}$. Then write $k'_a = \frac{K}{\eta}$.

As in [10], choose $K = 0.1$, $q_0 = 9$ and $\eta = 2$. Hence, by replacing these values into (42), one can find the system scalars as $\mathbf{A} = \frac{1}{\eta} = -\frac{1}{2}$, $\mathbf{B} = q_0 k'_a = \frac{9}{20}$, $\mathbf{N} = -k'_a = -\frac{1}{20}$. Finally, let $\mathbf{C} = 1$ (the observed output coincides with $q(t)$).

Let $\mu_1 = \frac{1}{4}j$ and $\mu_2 = -\frac{1}{4}j$ be the left sampling points. Additionally, let $\lambda_1 = \frac{1}{2}j$ and $\lambda_2 = -\frac{1}{2}j$ be the right sampling points. From (12) and (13), we put together the matrices as

$$\mathbb{L} = \begin{bmatrix} -\frac{9}{25} + \frac{27j}{25} & -\frac{27}{25} - \frac{9j}{25} \\ -\frac{27}{25} + \frac{9j}{25} & -\frac{9}{25} - \frac{27j}{25} \end{bmatrix}, \quad \mathbb{L}_s = \begin{bmatrix} \frac{9}{50} - \frac{27j}{50} & \frac{27}{50} + \frac{9j}{50} \\ \frac{27}{50} - \frac{9j}{50} & \frac{9}{50} + \frac{27j}{50} \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} \frac{18}{25} - \frac{9j}{25} \\ \frac{18}{25} + \frac{9j}{25} \end{bmatrix}, \quad \mathbf{W}^* = \begin{bmatrix} \frac{9}{20} + \frac{9j}{20} \\ \frac{9}{20} - \frac{9j}{20} \end{bmatrix}$$

We observe that $\text{rank}(\mathbb{L}) = 1$ and hence the dimension of the underlying system is $r = 1$. We choose projector vectors $\mathbf{x}^* = \mathbf{y} = \begin{bmatrix} j & -j \end{bmatrix}$. Then, the recovered values characterizing the dynamics can be found as

$$\hat{\mathbf{A}} = -\frac{1}{2}, \quad \hat{\mathbf{B}} = \frac{1}{2}, \quad \hat{\mathbf{C}} = \frac{9}{10}. \quad (45)$$

Then, using only one sample of the second transfer function \mathbf{H}_2 , for example $\mathbf{H}_2(\frac{1}{2}j) = \hat{\mathbf{C}}\hat{\Phi}(j)\hat{\mathbf{N}}\hat{\Phi}(\frac{1}{2}j)\hat{\mathbf{B}}$, one can recover the bilinear term $\hat{\mathbf{N}} = -\frac{1}{20}$. Note that the original scalar system is hence recovered ($\hat{\mathbf{A}} = \mathbf{A}$, $\hat{\mathbf{N}} = \mathbf{N}$, and $\hat{\mathbf{B}}\hat{\mathbf{C}} = \mathbf{B}\mathbf{C}$).

6.2 The Duffing oscillator

For the next example, we consider a single degree-of-freedom Duffing oscillator under harmonic input. It is one of the prototype systems of nonlinear dynamics and it became popular for studying anharmonic oscillations and, later, chaotic nonlinear dynamics. It is a non-linear second-order differential equation used to model certain damped and driven oscillators.

In the current example, the nonlinearity is characterized by quadratic and cubic stiffness as follows (see [8]):

$$m\ddot{x}(t) + c\dot{x}(t) + k_1x(t) + k_2x^2(t) + k_3x^3(t) = u(t), \quad (46)$$

where m is the mass coefficient, c is the viscous damping, k_1 is the linear damping, k_2 and k_3 are the quadratic and, respectively, cubic stiffness constants. Traditionally, as noted in [13], the Duffing equation does not have a quadratic term, i.e. $k_2 = 0$ in (46).

Additionally, in (46), the values \ddot{x} , \dot{x} and x represent the acceleration, velocity and displacement, respectively. The input is chosen to be $u(t) = Ae^{j\omega t}$, where A is the forcing level of amplitude and ω the frequency of excitation.

Now, by means of the harmonic probing procedure, one can obtain expressions for the transfer functions. By following the procedure described in [8], the first three transfer functions can be derived as follows (using the control input $u(t) = Ae^{j\omega t}$):

$$\begin{aligned} \mathbf{H}_1(j\omega) &= \frac{1}{-m\omega^2 + cj\omega + k_1}, \quad \mathbf{H}_2(j\omega, j\omega) = -k_2\mathbf{H}_1(j\omega)^2\mathbf{H}_1(2j\omega), \\ \mathbf{H}_3(j\omega, j\omega, j\omega) &= 2k_2^2\mathbf{H}_1^3(j\omega)\mathbf{H}_1(2j\omega)\mathbf{H}_1(3j\omega) - k_3\mathbf{H}_1^3(j\omega)\mathbf{H}_1(3j\omega). \end{aligned} \quad (47)$$

Next, we will introduce additional surrogate state variables in order to rewrite the second-order cubic differential equation in (46) as a first-order quadratic equation.

The case with $k_3 = 0$ In this case, the equation in (46) can be equivalently rewritten as:

$$\ddot{x}(t) = -\frac{c}{m}\dot{x}(t) - \frac{k_1}{m}x(t) - \frac{k_2}{m}x^2(t) + \frac{1}{m}u(t). \quad (48)$$

Introduce the following augmented state variable $\mathbf{z} = [x, \dot{x}]^T \in \mathbb{R}^2$ by putting together the original variable $x(t)$ and its derivative $\dot{x}(t)$. By doing this, we can rewrite (48) as follows:

$$\underbrace{\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}}_{\dot{\mathbf{z}}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k_1}{m} & -\frac{c}{m} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x \\ \dot{x} \end{bmatrix}}_{\mathbf{z}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{k_2}{m} & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{Q}} \underbrace{\begin{bmatrix} x \\ \dot{x} \end{bmatrix} \otimes \begin{bmatrix} x \\ \dot{x} \end{bmatrix}}_{\mathbf{z} \otimes \mathbf{z}} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{\mathbf{B}} u, \quad (49)$$

or in short $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{Q}(\mathbf{z} \otimes \mathbf{z}) + \mathbf{B}u$, a quadratic system. Choose to observe the original variable x and introduce the observed output $y(t) = x(t)$. Hence write $y = \mathbf{C}\mathbf{z}$, where $\mathbf{C} = [1 \ 0]$.

Choose the following parameters $m = 2$, $c = 10$, $k_1 = 10^2$ and $k_2 = 10^2$.

Let $\mu_1 = j$ and $\mu_2 = -j$ be the left sampling points. Additionally, let $\lambda_1 = 2j$ and $\lambda_2 = -2j$ be the right sampling points. From (12) and (13), we put together the matrices \mathbb{L}, \mathbb{L}_s and vectors \mathbf{V}, \mathbf{W} . These matrices have complex entries. By using the matrix $\mathbf{J} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix}$, we compute a real-valued equivalent realization: $\tilde{\mathbb{L}} = \mathbf{J}\mathbb{L}\mathbf{J}^*$, $\tilde{\mathbb{L}}_s = \mathbf{J}\mathbb{L}_s\mathbf{J}^*$, $\tilde{\mathbf{V}} = \mathbf{J}\mathbf{V}$, $\tilde{\mathbf{W}} = \mathbf{W}\mathbf{J}^*$. By incorporating the $\hat{\mathbf{E}}$ matrix into the others, we identify the three matrices of the linear model as

$$\hat{\mathbf{A}} = \begin{bmatrix} -5 & -2 \\ 25 & 0 \end{bmatrix}, \quad \hat{\mathbf{B}} = \begin{bmatrix} 3.5355 \\ -16.2635 \end{bmatrix}, \quad \hat{\mathbf{C}} = [0.0147 \ 0.0032]. \quad (50)$$

Next, in order to recover the quadratic term $\hat{\mathbf{Q}}$, we use the samples of the second transfer function $\mathbf{H}_2(j\omega)$ for $\omega \in \{\pm j, \pm 2j\}$. Then, we explicitly write the vector $\mathbf{V}_2 \in \mathbb{C}^4$ as

$$\mathbf{V}_2 = [\mathbf{H}_2(j) \ ; \ \mathbf{H}_2(-j) \ ; \ \mathbf{H}_2(2j) \ ; \ \mathbf{H}_2(-2j)]. \quad (51)$$

Additionally, we put together the matrix $\mathbf{T} \in \mathbb{C}^{4 \times 8}$ based on the recovered matrices in (50) as described in Section 5.2. Finally, the matrix $\mathbf{Q} \in \mathbb{R}^{2 \times 4}$

$$\hat{\mathbf{Q}} = \begin{bmatrix} -0.1111 & -0.0526 & -0.0526 & -0.0339 \\ 0.0462 & 0.0549 & 0.0549 & 0.0964 \end{bmatrix}.$$

Note that the lifted quadratic system in (49) is hence perfectly recovered. The system realizations $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{Q})$ and $(\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}, \hat{\mathbf{Q}})$ are equivalent (they only differ through a similarity transformation).

The case with $k_3 \neq 0$ In this case, the equation in (46) can be equivalently rewritten as:

$$\ddot{x}(t) = -\frac{c}{m}\dot{x}(t) - \frac{k_1}{m}x(t) - \frac{k_2}{m}x^2(t) - \frac{k_3}{m}x^3(t) + \frac{1}{m}u(t). \quad (52)$$

Introduce the new augmented state variable $\mathbf{w} = [x, \dot{x}, x^2]^T \in \mathbb{R}^3$ by also including the squared variable x . By doing this, we can rewrite (52) as follows:

$$\underbrace{\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{x}^2 \end{bmatrix}}_{\dot{\mathbf{w}}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -\frac{k_1}{m} & -\frac{c}{m} & -\frac{k_2}{m} \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x \\ \dot{x} \\ x^2 \end{bmatrix}}_{\mathbf{w}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{k_3}{m} & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{Q}} \underbrace{\begin{bmatrix} x \\ \dot{x} \\ x^2 \end{bmatrix}}_{\mathbf{w}} \otimes \underbrace{\begin{bmatrix} x \\ \dot{x} \\ x^2 \end{bmatrix}}_{\mathbf{w}} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \\ 0 \end{bmatrix}}_{\mathbf{B}} u,$$

or in short $\dot{\mathbf{w}} = \mathbf{A}\mathbf{w} + \mathbf{Q}(\mathbf{w} \otimes \mathbf{w}) + \mathbf{B}u$. As before, we introduce the observed output $y(t) = x(t)$. Hence write $y = \mathbf{C}\mathbf{w}$, where $\mathbf{C} = [1 \ 0 \ 0]$. Next, choose the same parameters as for the previous case and, additionally, take $k_3 = 10^4$.

Let $\mu_1 = 1$, $\mu_2 = j$ and $\mu_3 = -j$ be the left sampling points. Additionally, let $\lambda_1 = 2$, $\lambda_2 = 2j$ and $\lambda_3 = -2j$ be the right sampling points. From (12) and (13), we put together a Loewner model of dimension $k = 3$ (with complex entries). After transforming the matrices into real arithmetics, we note that $\text{rank}(\mathbb{L}) = 2$. Based on the procedure in (18), we compress the dimension of the model to $r = 2$ (by means of left and right singular vector projection matrices). For the new compressed real-valued model, the Loewner pencil is regular. Note that the eigenvalues of this pencil are $-\frac{5}{2} \pm \frac{5}{2}\sqrt{7}$, exactly as the ones of the original matrix \mathbf{A} . Moreover, these values are the roots of the denominator of $\mathbf{H}_1(s)$ in (47), i.e. $ms^2 + cs + k_1 = 0$.

We choose the input to be $u(t) = \cos(4t)$ and simulate the original cubic scalar system in (52) for zero initial conditions. Additionally, we also perform a time-domain simulation of the reduced Loewner model. The time horizon is chosen to be $[0, 10]$ s. The first order derivative is approximated via a Runge-Kutta 4th/5h order scheme (*ode45* in MATLAB®).

We observe as before the displacement variable, i.e. the output is chosen to be $y = x$. The results are depicted presented in Fig. 1. Note that the Loewner model provides good approximation to the original cubic system.

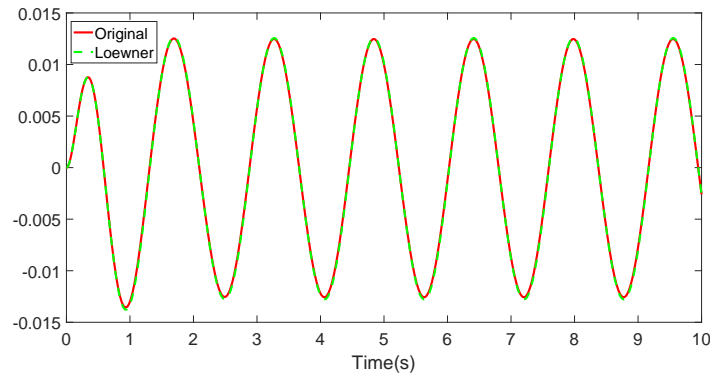


Figure 1: Time-domain simulation of the observed output: the displacement

7 CONCLUSIONS

In this study, we analyze a method that can use experimental data for identifying some classes of nonlinear systems (bilinear and quadratic). A linear model is first fitted from measurements of the first (linear) transfer function. Then, depending on the desired class of nonlinearities, an extra term is added to the differential equation to be modeled. The first numerical example is bilinear while the second has a cubic nonlinearity. For the later, a reformulation as a quadratic system is first enforced. The test cases show promising numerical results. Further research topics include applying this approach to quadratic-bilinear systems and to systems with higher dimension (larger number of degrees of freedom).

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